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SELF-SIMILAR SETS OF ZERO HAUSDORFF MEASURE AND POSITIVE PACKING MEASURE

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ABSTRACT

We prove that there exist self-similar sets of zero Hausdorff measure, but positive and finite packing measure, in their dimension; for instance, for almost every $u \in [3,6]$, the set of all sums $\sum_0^\infty a_n 4^{-n}$ with digits with $a_n \in \{0,1,u\}$ has this property. Perhaps surprisingly, this behavior

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is typical in various families of self-similar sets, e.g., for projections of certain planar self-similar sets to lines. We establish the Hausdorff measure result using special properties of self-similar sets, but the result on packing measure is obtained from a general complement to Marstrand's projection theorem, that relates the Hausdorff measure of an arbitrary Borel set to the packing measure of its projections.

1. Introduction

Self-similar sets, i.e., compact sets $\mathcal{K}$ that satisfy $\mathcal{K} = \bigcup_i S_i\mathcal{K}$ for some finite collection of contracting similitudes $\{S_i\}$, are well understood under separation conditions, such as the "Open Set Condition" (OSC, see [9]), but they remain quite mysterious when arbitrary overlaps are permitted. In general it is known that the Hausdorff, Minkowski and packing dimensions of a self-similar set coincide [4], but much less is known about the behavior of Hausdorff and packing measures on general self-similar sets. We prove

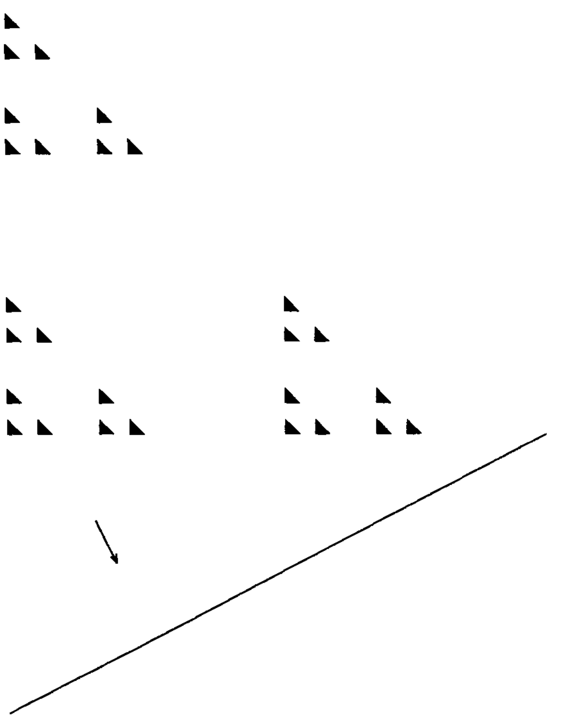

Figure 1. Projecting an s-dimensional Sierpinski gasket for $s < 1$.

THEOREM 1.1: *There exist self-similar sets in $\mathbb{R}$ that have zero Hausdorff measure but positive and finite packing measure in their dimension. In particular, if $\frac{1}{5} < r < \frac{1}{3}$, then this holds for the self-similar sets*

$$\mathcal{K}_u^r := \Big\{ \sum_{n=0}^{\infty} a_n r^n \colon a_n \in \{0, 1, u\} \Big\}$$

for a.e. u in a certain nonempty interval.

Up to scaling, $\mathcal{K}_u^r$ can be identified with the orthogonal projection of the s-dimensional Sierpinski gasket $\mathcal{G}^r$ (where $s = s(r) = \log 3/|\log r|$) on the line $y = ux$, see Figure 1.

Other dynamical settings where it has been shown that the natural measures are packing measures rather than Hausdorff measures are limit sets of certain Kleinian groups [26] and parabolic Julia sets [3]; in these cases the phenomenon is due to parabolic fixed points. Self-similar sets have a more rigid structure, and it seems much harder to construct an explicit example of a self-similar set as in the theorem. Indeed, no such example is known, and in particular it is an unsolved problem to exhibit specific parameters r, u for which the conclusion of Theorem 1.1 holds.

Next, we illustrate our results on families arising from projections of a given set in the plane. The i.f.s. $\{S_i\}_{i \le m}$ is said to satisfy the **Open Set Condition** (OSC) if there exists a non-empty open set U such that $S_i U$ are disjoint and lie in U for $i = 1, \ldots, m$. Consider a self-similar set $\mathcal{K} \subset \mathbb{R}^2$, defined as the unique non-empty compact satisfying

$$\mathcal{K} = \bigcup_{i=1}^{m} (r_i \mathcal{K} + b_i), \quad \text{with } r_i \in (0,1) \quad \text{and} \quad b_i \in \mathbb{R}^2 \tag{1.1}$$

(thus, rotations are not allowed). We assume that the similitudes $S_i(x) := r_i x + b_i$ for $i = 1, \ldots, m$ satisfy the OSC. It is well-known (see [9]) that the Hausdorff dimension $\dim_H \mathcal{K}$ equals the similarity dimension s, defined by $\sum_{i=1}^m r_i^s = 1$, and the s-dimensional Hausdorff measure $\mathcal{H}^s(\mathcal{K})$ is positive and finite. For any $\theta \in [0, \pi)$ the orthogonal projection of $\mathcal{K}$ on the line $y \cos\theta = x \sin\theta$, denoted $\mathrm{proj}_\theta \mathcal{K}$, is self-similar. If $s < 1$, then Marstrand's theorem (see [5]) says that $\dim_H(\mathrm{proj}_\theta \mathcal{K}) = s$ for Lebesgue-a.e. θ. Thus, it is natural to inquire about the s-dimensional measures of projections. To state our Hausdorff measure result, let

$$\mathcal{IP} = \big\{\theta \in [0, \pi) \colon \mathrm{proj}_\theta \colon \mathcal{K} \to \mathbb{R} \text{ is not one-to-one}\big\}$$

(the letters "$\mathcal{IP}$" stand for "intersection parameters"). In the second part of the theorem we assume the **Strong Separation Condition**, i.e., that $S_i(\mathcal{K}) \cap S_j(\mathcal{K}) = \emptyset$ for $i \neq j$.

THEOREM 1.2: *Let $\mathcal{K} \subset \mathbb{R}^2$ be a self-similar set (1.1) of dimension $s \in (0,1)$ that is not on a line.*

(i) *If the i.f.s. $\{S_i\}_{i \le m}$ satisfies the OSC, then $\mathcal{H}^s(\mathrm{proj}_\theta \mathcal{K}) = 0$ for Lebesgue-a.e. $\theta \in \mathcal{IP}$.*

(ii) *If the i.f.s. $\{S_i\}_{i \le m}$ satisfies the Strong Separation Condition, then $\mathcal{IP}$ is a compact perfect set, and the set $\{\theta \in \mathcal{IP}\colon \mathcal{H}^s(\mathrm{proj}_\theta \mathcal{K}) = 0\}$ is a dense G_δ set in $\mathcal{IP}$.*

Part (i) of the theorem has content only if $\mathcal{IP}$ has positive Lebesgue measure, but part (ii) is always meaningful. One case where the theorem applies was indicated in Theorem 1.1; see Example 2.8. As another example, consider $\mathcal{K} = \mathcal{C}_r \times \mathcal{C}_r$ where $\mathcal{C}_r$ is the standard middle-α Cantor set, with $\alpha = 1 - 2r$. We have $s = \dim_H \mathcal{K} = \log 4/|\log r|$, so $s < 1$ when $r < \frac{1}{4}$. We show in Example 6.1 that the set $\mathcal{IP} = \mathcal{IP}(r)$ contains a non-empty interval for $r \in (\frac{1}{6}, \frac{1}{4})$.

Next we state the result on packing measure $\mathcal{P}^s$, again restricting attention to projections. The method of proof can be traced to Kaufman [11].

PROPOSITION 1.3: *Let $\mathcal{K} \subset \mathbb{R}^2$ be any Souslin set such that $\mathcal{H}^s(\mathcal{K}) > 0$ for some $s \in (0,1)$. Then $\mathcal{P}^s(\mathrm{proj}_\theta \mathcal{K}) > 0$ for a.e. θ. Moreover,*

$$\dim_H \{\theta \in [0,\pi)\colon \mathcal{P}^s(\mathrm{proj}_\theta \mathcal{K}) = 0\} \le s.$$

Remarks: (a) Note that the assumption involves the Hausdorff measure of $\mathcal{K}$. It cannot be stated in terms of the packing measure of $\mathcal{K}$, since packing dimension may drop for almost all projections, see Järvenpää [10] and Falconer–Howroyd [6].

(b) Proposition 1.3, in conjunction with Theorem 1.2 and Example 2.8, yields Theorem 1.1.

(c) Proposition 1.3 is quite close to Mattila [15, Theorem 4.3], we derive it from a more general result (Theorem 4.1) that applies to parameterized families where the similarity dimension varies. For instance, this theorem implies that the set $\mathcal{K}_u^r = \left\{\sum_{n=0}^{\infty} a_n r^n\colon a_n \in \{0,1,u\}\right\}$ has positive packing measure in its dimension $s(r) = \log 3/|\log r|$, for every $u \in [2,4]$ and a.e. $r \in [\frac{1}{1+u}, \frac{1}{3}]$; see Example 2.9. This result cannot be obtained from a statement on orthogonal projections.

To prove the result on packing measure, we derive estimates that also determine which kernels assign positive capacity to typical projections.

Let $\Phi \in C^1(0,\infty)$ be a nonnegative decreasing function which vanishes for all x sufficiently large. Recall that for any Souslin set F, the capacity $\mathrm{Cap}_\Phi(F)$ is positive if and only if F supports a positive Borel measure such that $\int\int \Phi(|x-y|)\,d\nu(x)d\nu(y) < \infty$ (see [2]).

COROLLARY 1.4: *Let $\mathcal{K} \subset \mathbb{R}^2$ be a Souslin set, and let $s \in (0,1)$. Consider a kernel Φ as above.*

(i) *If $\mathcal{H}^s(\mathcal{K}) > 0$ and $\int_0^\infty r^s|\Phi'(r)|dr < \infty$, then*

$$\dim_H\{\theta\colon \mathrm{Cap}_\Phi(\mathrm{proj}_\theta \mathcal{K}) = 0\} \leq s.$$

(ii) *If $\mathcal{P}^s(\mathcal{K}) < \infty$ and $\int_0^\infty r^s|\Phi'(r)|dr = \infty$, then*

$$\mathrm{Cap}_\Phi(\mathrm{proj}_\theta \mathcal{K}) = 0 \quad \textit{for all } \theta \in [0,\pi).$$

(Part (ii) is easily derived, and is included for comparison.)

Background. Let $\mathcal{K}$ be a self-similar set on the real line. Schief [23], building on the work of Bandt and Graf [1], showed that $\mathcal{H}^s(\mathcal{K}) > 0$ is equivalent to the OSC. In many interesting examples it appears that the iterated function system has an "overlap"; however, it is non-trivial to verify that the OSC fails since the open set U may be rather complicated. Pollicott and Simon [22] considered families defined by λ-expansions with deleted digits, with the contraction ratio λ as parameter. They proved that in many cases the Hausdorff dimension coincides with the similarity dimension for almost every λ in some interval J, in spite of an apparent overlap. Solomyak [25] showed that the self-similar sets studied in [22] have zero Hausdorff measure in their dimension for a.e. $\lambda \in J$. The proof used in an essential way arithmetic properties of those sets and did not extend even to the simplest families of projections.

The rest of the paper is organized as follows. Section 2 contains general statements of results for one-dimensional self-similar sets and examples. The proofs of Hausdorff measure results are in Section 3. In Section 4 we prove a general result on packing measure in the setting of a family of maps from a metric space to $\mathbb{R}$. Capacities in general kernels are considered in Section 5 where Corollary 1.4 is derived. Further examples are collected in Section 6. Section 7 contains generalizations to higher dimensions, concluding remarks, and open questions. The reader is referred to the books [5, 16] for background in dimension theory and material concerning self-similar sets.

2. Self-similar sets

Consider a one-parameter family of iterated function systems (i.f.s.) $\{S_1^\lambda, \dots, S_m^\lambda\}_{\lambda \in J}$ where

$$S_i^\lambda(x) = r_i(\lambda)x + a_i(\lambda) \quad \text{for } x \in \mathbb{R},$$

and $J \subset \mathbb{R}$ is a closed interval. We assume that $a_i(\lambda)$ and $r_i(\lambda)$ belong to $C^1(J)$ and

$$0 < \beta \le r_i(\lambda) \le \rho < 1 \quad \text{for all } i \le m \quad \text{and} \quad \lambda \in J. \tag{2.1}$$

Let $\mathcal{K}^\lambda$ be the self-similar set corresponding to λ, that is, $\mathcal{K}^\lambda = \bigcup_{i=1}^m \mathcal{K}_i^\lambda$ where $\mathcal{K}_i^\lambda = S_i^\lambda(\mathcal{K}^\lambda)$. The similarity dimension $s(\lambda)$ is the unique solution of the equation $\sum_{i=1}^m r_i^{s(\lambda)}(\lambda) = 1$. If $r_i(\lambda) = r(\lambda)$ for all $i \le m$, we say that the i.f.s. $\{S_i^\lambda\}_{i \le m}$ is **homogeneous**; then $s(\lambda) = \log m / |\log r(\lambda)|$. Denote $\mathcal{A} = \{1, \dots, m\}$ and $\Omega = \mathcal{A}^{\mathbb{N}}$. For $u \in \mathcal{A}^n$ we write $S_u^\lambda = S_{u_1}^\lambda \circ \cdots \circ S_{u_n}^\lambda$ and $r_u(\lambda) = r_{u_1}(\lambda) \cdot \ldots \cdot r_{u_n}(\lambda)$. The map $\Pi(\lambda, \cdot)\colon \Omega \to \mathcal{K}^\lambda$ defined by

$$\Pi(\lambda, \omega) = \lim_{n \to \infty} S_{\omega_1 \dots \omega_n}^\lambda(0) = \sum_{n=1}^{\infty} r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda) \tag{2.2}$$

will be called the "natural projection map" below. It follows from (2.1) and (2.2) that $\Pi(\cdot, \omega) \in C^1(J)$ for all $\omega \in \Omega$, and moreover,

$$\text{the map } \omega \mapsto \Pi(\cdot, \omega) \text{ is continuous from } \Omega \text{ to } C^1(J). \tag{2.3}$$

Denote

$$f_{\omega,\tau}(\lambda) = \Pi(\lambda, \omega) - \Pi(\lambda, \tau).$$

We say that the **transversality condition** holds on J if for any $\omega, \tau \in \Omega$,

$$\text{if } \exists \lambda \in J\colon f_{\omega,\tau}(\lambda) = f'_{\omega,\tau}(\lambda) = 0 \quad \text{then} \quad f_{\omega,\tau} \equiv 0. \tag{2.4}$$

Define

$$\mathcal{IP} := \big\{\lambda \in J\colon \exists \omega, \tau \in \Omega\colon f_{\omega,\tau}(\lambda) = 0 \text{ but } f_{\omega,\tau} \not\equiv 0\big\}. \tag{2.5}$$

When we considered, in Theorem 1.2, the projections of a planar self-similar set $\mathcal{K}$, we assumed that $\mathcal{K}$ is not on a line and (in part (ii)) that it satisfies the Strong Separation Condition. The analogues of these assumptions in the current general setting are

$$\forall \lambda \in J, \quad \mathcal{K}^\lambda \text{ is not a singleton} \tag{2.6}$$

and

$$f_{\omega,\tau} \equiv 0 \Rightarrow \omega = \tau. \tag{2.7}$$

THEOREM 2.1: *Suppose that the one-parameter family of iterated function systems $\{S_1^\lambda, \ldots, S_m^\lambda\}_{\lambda \in J}$ satisfies (2.1), (2.4) and $\mathcal{IP} \neq \emptyset$. Then*

(i) $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0$ *for Lebesgue-a.e.* $\lambda \in \mathcal{IP}$.

(ii) *Suppose, in addition, that (2.6) and (2.7) hold. Then $\mathcal{IP}$ is a compact perfect set, and the set $\{\lambda \in \mathcal{IP}\colon \mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0\}$ is a dense G_δ set in $\mathcal{IP}$.*

(iii) *Suppose that (2.1), (2.4) and (2.7) hold, and moreover, $r_i(\lambda) = r_i^{\varphi(\lambda)}$ for some positive function $\varphi(\lambda)$ and some reals $r_i \in (0,1)$. Then $0 < \mathcal{P}^{s(\lambda)}(\mathcal{K}^\lambda) < \infty$ for all $\lambda \in J$ except a set of Hausdorff dimension $s_{\max} = \sup_{\lambda \in J} s(\lambda)$.*

Remarks: (a) Parts (i) and (ii) are proved in Section 3; part (iii) is derived in Section 4 from a more general theorem which has nothing to do with self-similarity.

(b) Part (i) of the theorem has content whenever the set $\mathcal{IP}$ has positive Lebesgue measure. We discuss below how to check this.

(c) The condition on contraction rates $r_i(\lambda)$ in part (iii) is satisfied, e.g., when $\{S_i^\lambda\}_{i \le m}$ is a homogeneous i.f.s. for all $\lambda \in J$, or when $r_i(\lambda) \equiv r_i$.

Given an i.f.s. of similitudes $\{S_i\}_{i \le m}$ and a probability vector $\{p_i\}_{i \le m}$, the corresponding **self-similar measure** is defined as the unique Borel probability measure ν satisfying $\nu = \sum_{i \le m} p_i \nu \circ S_i^{-1}$; see Hutchinson [9]. Of course, ν is supported on the self-similar set for the i.f.s. The self-similar measure is called **natural** if $p_i = r_i^s$ where s is the similarity dimension.

COROLLARY 2.2: *For any self-similar set $\mathcal{K}$ with similarity dimension s, if $0 < \mathcal{P}^s(\mathcal{K}) < \infty$, then $\mathcal{P}^s|_{\mathcal{K}}$ is the natural self-similar measure up to scaling (the same is true for the Hausdorff measure). In particular, under the conditions of Theorem 2.1(iii), for all $\lambda \in J$ except a set of Hausdorff dimension $s_{\max}$, the normalized restriction of $\mathcal{P}^{s(\lambda)}$ to $\mathcal{K}^\lambda$ coincides with the natural self-similar measure on $\mathcal{K}^\lambda$.*

Proof: The argument is analogous to that in [9, 5.3(1)(iii)], but we provide it for completeness. Suppose that $\mathcal{K} = \bigcup_{i \le m}(S_i\mathcal{K}) = \bigcup_{i \le m} \mathcal{K}_i$ and S_i has contraction factor r_i. By subadditivity,

$$\mathcal{P}^s(\mathcal{K}) \le \sum_{i \le m} \mathcal{P}^s(\mathcal{K}_i) = \sum_{i \le m} r_i^s \mathcal{P}^s(\mathcal{K}) = \mathcal{P}^s(\mathcal{K}).$$

Thus, the inequality above is an equality; since $0 < \mathcal{P}^s(\mathcal{K}) < \infty$, we must have $\mathcal{P}^s(\mathcal{K}_i \cap \mathcal{K}_j) = 0$ for $i \neq j$. Therefore, for any Borel set $A \subset \mathcal{K}$,

$$\begin{aligned}\mathcal{P}^s(A) = \sum_{i\le m} \mathcal{P}^s(A \cap \mathcal{K}_i) &= \sum_{i \le m} \mathcal{P}^s(S_i(S_i^{-1}A \cap \mathcal{K})) \\ &= \sum_{i\le m} r_i^s \mathcal{P}^s(S_i^{-1}A \cap \mathcal{K}).\end{aligned}$$

This shows that the restriction of $\mathcal{P}^s$ to $\mathcal{K}$ satisfies the equation defining the natural self-similar measure, and the claim follows by its uniqueness. ∎

Next we give some useful consequences of (2.7). We note that (2.7) holds in all natural applications, except in the case of projections of a self-similar set satisfying the OSC but without Strong Separation.

LEMMA 2.3: *Suppose that (2.7) holds. Then*

(i) $\mathcal{IP} = \{\lambda \in J\colon \exists i \neq j,\ \mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda \neq \emptyset\}$.

(ii) $\mathcal{IP}$ *is compact.*

(iii) *If* $\lambda \notin \mathcal{IP}$, *then* $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) > 0$.

(iv) *If (2.4) holds on* J, *then there exists* $\delta > 0$ *such that for all* $\omega, \tau \in \Omega$ *with* $\omega_1 \neq \tau_1$,

$$\lambda \in J,\ |f_{\omega,\tau}(\lambda)| < \delta \Rightarrow |f'_{\omega,\tau}(\lambda)| > \delta. \tag{2.8}$$

Proof: (i) If $\lambda \in \mathcal{IP}$, then there exist $\omega \neq \tau$ such that $f_{\omega,\tau}(\lambda) = 0$. Using that $\Pi(\lambda, \alpha\omega) = S_\alpha \Pi(\lambda, \omega)$ for all $\alpha \in \{1, \ldots, m\}$, and the invertibility of S_α, we find ω', τ' with $i = \omega_1' \neq \tau_1' = j$ such that $f_{\omega',\tau'}(\lambda) = 0$, whence $\mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda \neq \emptyset$. The other direction follows from (2.7).

(ii) By part (i), $\lambda_0 \in \text{clos}(\mathcal{IP})$ implies that there are $\lambda_n \to \lambda_0$ such that $f_{\omega^{(n)},\tau^{(n)}}(\lambda_n) = 0$ for some $\omega^{(n)}$ and $\tau^{(n)}$ with $\omega_1^{(n)} \neq \tau_1^{(n)}$. Using compactness of Ω we obtain that $\lambda_0 \in \mathcal{IP}$.

(iii) If $\lambda \notin \mathcal{IP}$ then, by part (i), the i.f.s. $\{S_i^\lambda\}_{i \le m}$ satisfies the Strong Separation Condition and hence $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) > 0$.

(iv) By (2.3), the set $\{f_{\omega,\tau}\colon \omega_1 \neq \tau_1\}$ is compact in $C^1(J)$. If (2.8) is false, then the compactness argument shows that there are ω, τ with $\omega_1 \neq \tau_1$, and $\lambda \in J$, such that $f_{\omega,\tau}(\lambda) = f'_{\omega,\tau}(\lambda) = 0$. In view of (2.3), this is a contradiction. ∎

The measure of the set of intersection parameters. Let $\mathcal{L}$ denote Lebesgue measure on the line. The following proposition contains a necessary condition for $\mathcal{L}(\mathcal{IP}) > 0$; it is proved in Section 3.

PROPOSITION 2.4: *Suppose that the i.f.s. satisfies conditions (2.1) and (2.4), as in Theorem 2.1(i). Then* $\dim_H \mathcal{IP} \le 2s_{\max}$; *thus, if* $s_{\max} \in (0, \frac{1}{2})$ *then* $\mathcal{L}(\mathcal{IP}) = 0$.

We have an easy-to-check (although far from sharp) sufficient condition for $\mathcal{L}(\mathcal{IP}) > 0$ only in the homogeneous case. Some non-homogeneous families can be handled as well, see Example 6.2. Denote by $\mathsf{Conv}(A)$ the convex hull of a set A.

LEMMA 2.5: *Let* $\{S_1^\lambda, \ldots, S_m^\lambda\}_{\lambda \in J}$ *be a one-parameter family of homogeneous i.f.s. satisfying (2.7). Suppose that for some* $i \neq j$ *there exists a subinterval* $\widetilde{J} \subset J$ *such that*

$$\mathsf{Conv}(\mathcal{K}_i^\lambda) \cap \mathsf{Conv}(\mathcal{K}_j^\lambda) \neq \emptyset \quad \textit{for all } \lambda \in \widetilde{J} \tag{2.9}$$

and

$$\mathcal{K}^\lambda - \mathcal{K}^\lambda \textit{ is an interval for all } \lambda \in \widetilde{J}. \tag{2.10}$$

Then $\widetilde{J} \subset \mathcal{IP}$.

Proof: Condition (2.10) implies $\mathcal{K}^\lambda - \mathcal{K}^\lambda = \mathsf{Conv}(\mathcal{K}^\lambda) - \mathsf{Conv}(\mathcal{K}^\lambda)$. For any set $A \subset \mathbb{R}$ we have $A - A = \{t \in \mathbb{R}\colon A \cap (A+t) \neq \emptyset\}$. Since the i.f.s. are homogeneous, $\mathcal{K}_i^\lambda$ and $\mathcal{K}_j^\lambda$ are both translates of $r(\lambda)\mathcal{K}^\lambda$; thus, (2.9) implies $\mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda \neq \emptyset$ for all $\lambda \in \widetilde{J}$, and the lemma follows by Lemma 2.3. ∎

Remark: Conditions (2.9) and (2.10) are easy to verify. Indeed, let $\mathcal{K}$ be the self-similar set for a homogeneous i.f.s. $\{S_i\}_{i \le m}$, with $S_i(x) = rx + a_i$. Then $\mathcal{K} - \mathcal{K}$ is also a self-similar set:

$$\mathcal{K} - \mathcal{K} = \left\{ \sum_{n=0}^{\infty} c_n r^n \colon c_n \in \Gamma \right\}$$

where $\Gamma = \{a_i - a_j\}_{i,j \le m}$. Let g be the minimal gap between two consecutive elements of $\{a_i\}_{i \le m}$ and let G be the maximal gap between two consecutive elements of Γ. It is easy to see that $\mathsf{Conv}(\mathcal{K}_i) \cap \mathsf{Conv}(\mathcal{K}_j) \neq \emptyset$ for some $i \neq j$ if and only if

$$\frac{r}{1-r} \max \Gamma \ge g, \tag{2.11}$$

and $\mathcal{K} - \mathcal{K}$ is an interval if and only if

$$\frac{r}{1-r} \max \Gamma \ge \frac{1}{2} G. \tag{2.12}$$

Families of projections. Let $\mathcal{K} = \bigcup_{i=1}^m (r_i\mathcal{K} + b_i) \subset \mathbb{R}^2$ be a self-similar set (1.1). We are interested in the one-parameter family of projections $\{\mathrm{proj}_\theta \mathcal{K}\}_{\theta\in[0,\pi)}$. The corresponding similitudes are $S_i^\theta(x) = r_i x + \mathrm{proj}_\theta b_i$.

LEMMA 2.6: *The one-parameter family of i.f.s. $\{S_1^\theta, \ldots, S_m^\theta\}_{\theta\in[0,\pi)}$ satisfies the transversality condition (2.4).*

Proof: Let $\widetilde{\Pi}\colon \Omega \to \mathcal{K}$ be the natural projection map. We have

$$\Pi(\theta,\omega) = \underset{\theta}{\mathrm{proj}} \circ \widetilde{\Pi}(\omega) = \widetilde{\Pi}(\omega)\cdot(\cos\theta, \sin\theta),$$

hence

$$|\Pi(\theta,\omega) - \Pi(\theta,\tau)|^2 + \left|\frac{d}{d\theta}\big(\Pi(\theta,\omega) - \Pi(\theta,\tau)\big)\right|^2 = |\widetilde{\Pi}(\omega) - \widetilde{\Pi}(\tau)|^2.$$

Since $f_{\omega,\tau}(\theta) \equiv 0$ if and only if $\widetilde{\Pi}(\omega) = \widetilde{\Pi}(\tau)$, the condition (2.4) follows. ∎

Remarks: (a) For a family of projections, $\theta \in \mathcal{IP}$ iff $\mathrm{proj}_\theta : \mathcal{K} \to \mathbb{R}$ is not one-to-one. Thus, Theorem 2.1(i)(ii), combined with Lemma 2.6, immediately implies Theorem 1.2.

(b) There is an equivalent way to represent families of projections which is sometimes convenient. Let $S_i^\lambda(x) = r_i x + (c_i + d_i\lambda)$ for $i \le m$. Then $\mathcal{K}^\lambda = \sqrt{1+\lambda^2}\,\mathrm{proj}_\theta \widetilde{\mathcal{K}}$ where $\tan\theta = \lambda$ and $\widetilde{\mathcal{K}}$ is the limit set of the planar i.f.s. $\{(x,y) \mapsto r_i(x,y) + (c_i,d_i)\}_{i\le m}$.

The next lemma sharpens Proposition 2.4 for families of projections.

LEMMA 2.7: *Let $\mathcal{K} \subset \mathbb{R}^2$ be a self-similar set (1.1) that satisfies the Strong Separation Condition. Then $\dim_H \mathcal{IP} \le \dim_H(\mathcal{K} - \mathcal{K})$. Therefore, if $\dim_H(\mathcal{K} - \mathcal{K}) < 1$, then $\mathcal{H}^s(\mathrm{proj}_\theta \mathcal{K}) > 0$ for a.e. θ; in particular, this is the case if $s = \dim_H \mathcal{K} < \frac{1}{2}$.*

Proof: The Strong Separation Condition implies that there exists $\eta > 0$ such that $\bigcup_{i\ne j}(\mathcal{K}_i - \mathcal{K}_j) \subset \{\mathbf{x} \in \mathbb{R}^2\colon |\mathbf{x}| \ge \eta\}$; note that $\Upsilon(\mathbf{x}) := \mathbf{x}/|\mathbf{x}|$ is Lipschitz on the latter set. By Strong Separation, the family of projections satisfies (2.7). Hence Lemma 2.3(i) implies that

$$\mathcal{IP} = \Big\{\theta \in [0,\pi)\colon (\cos\theta, \sin\theta) \in \Upsilon\Big(\bigcup_{i\ne j}(\mathcal{K}_i - \mathcal{K}_j)\Big)\Big\}.$$

Thus, $\dim_H \mathcal{IP} \le \max_{i,j} \dim_H(\mathcal{K}_i - \mathcal{K}_j) \le \dim_H(\mathcal{K} - \mathcal{K})$. The last claim follows by Lemma 2.3(iii). ∎

The following example was already mentioned in Theorem 1.1.

Example 2.8: Let

$$\mathcal{K}_u^r = \Big\{\sum_{n=0}^{\infty} a_n r^n \colon a_n \in \{0,1,u\}\Big\}. \tag{2.13}$$

We fix r and let $u \in \mathbb{R}$ be the parameter. According to the remark above, $\{\mathcal{K}_u^r\}_{u\in\mathbb{R}}$ is affine-equivalent to the family of projections of the s-dimensional Sierpinski gasket $\mathcal{G}^r = \Big\{\sum_{n=0}^{\infty} c_n r^n \colon c_n \in \{0,1,i\}\Big\}$, where $s = \log 3/|\log r|$. Thus, the transversality condition (2.4) holds by Lemma 2.6. The property $\mathcal{L}(\mathcal{IP}) > 0$ is checked with the help of Lemma 2.5. We have $\Gamma = \{0, \pm 1, \pm u, \pm(u-1)\}$. Assume, without loss of generality, that $u \geq 2$. Then $\max \Gamma = u$, $g = 1$, and $G = \max\{1, u-2\}$. It follows from (2.11) and (2.12) that $\mathcal{IP} \supset [\frac{1-r}{r}, \frac{2(1-r)}{1-3r}]$. This interval is nonempty for $r \in (\frac{1}{5}, \frac{1}{3})$. Theorem 2.1 implies that $\mathcal{H}^s(\mathcal{K}_u^r) = 0$ and $0 < \mathcal{P}^s(\mathcal{K}_u^r) < \infty$ for all $r \in (\frac{1}{5}, \frac{1}{3})$ and a.e. $u \in [\frac{1-r}{r}, \frac{2(1-r)}{1-r}]$. Note that $\mathcal{G}^r$ satisfies the Strong Separation Condition and $\mathcal{G}^r - \mathcal{G}^r$ is self-similar with similarity dimension $\log 7/|\log r|$. Thus, by Lemma 2.7, $\mathcal{H}^s(\mathcal{K}_u^r) > 0$ for all $r < \frac{1}{7}$ and a.e. u. The case $r \in (\frac{1}{7}, \frac{1}{5})$ remains open although we suspect that $\mathcal{L}(\mathcal{IP}) > 0$ for a.e. such r.

Families with contraction rate as parameter. Suppose that $S_i^\lambda(x) = \lambda x + a_i$ for $i \leq m$. The self-similar set for this i.f.s. is

$$\mathcal{K}^\lambda = \mathcal{K}^\lambda(D) = \Big\{\sum_{n=0}^{\infty} c_n \lambda^n \colon c_n \in D\Big\} \quad \text{where } D = \{a_i\}_{i\leq m}.$$

The study of such a family with the simplest apparent "overlap", for $D = \{0,1,3\}$ and $\lambda > \frac{1}{4}$, was initiated by Keane and Smorodinsky, see [12]. Pollicott and Simon [22] introduced the transversality condition and verified it in several cases. A more efficient method to check transversality was found by Solomyak [24] and refined by Peres and Solomyak [20]. The reader is referred to [21, 25] for a detailed discussion of this method and its consequences.

Example 2.9: Let $\mathcal{K}_u^r$ be as in (2.13) but, unlike Example 2.8, we fix u and use r as a parameter. Since we are concerned with sets of Hausdorff dimension less than one, $r < \frac{1}{3}$. Assume, without loss of generality, that $u \geq 2$. It follows from [21, Cor. 5.2(ii)] that if $u \in (2,4)$ then transversality in r holds for $r \in (0, \frac{1}{3})$. Lemma 2.5 implies (after a little calculation) that the property $\mathcal{L}(\mathcal{IP}) > 0$ holds in all these cases, with $\mathcal{IP} \supset [\frac{1}{1+u}, \frac{1}{3}]$. Theorem 2.1 implies that $\mathcal{H}^{s(r)}(\mathcal{K}_u^r) = 0$ and $0 < \mathcal{P}^{s(r)}(\mathcal{K}_u^r) < \infty$, with $s(r) = \log 3/|\log r|$, for all $u \in (2,4)$ and a.e. $r \in (\frac{1}{1+u}, \frac{1}{3})$.

Note that in the special case $u = 3$, Solomyak [25] showed $\mathcal{H}^{s(r)}(\mathcal{K}_u^r) = 0$ for a.e. $r \in (\frac{1}{4}, \frac{1}{3})$. However, the proof depended on the arithmetic nature of the digits and did not extend to other values of u.

3. Zero Hausdorff measure

Recall that $\mathcal{A} = \{1, \ldots, m\}$ and denote $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$. Write $|u| = n$ for $u \in \mathcal{A}^n$ and let $\omega|n = \omega_1 \ldots \omega_n$ for $\omega \in \Omega$. The proof of Theorem 2.1(i),(ii) uses the Bandt–Graf criterion for zero Hausdorff measure of a self-similar set. For $S\colon \mathbb{R}^d \to \mathbb{R}^d$ let $\|S\| = \sup\{|S\mathbf{x}|: |\mathbf{x}| \leq 1\}$.

THEOREM 3.1 (Bandt and Graf [1]): *Let $\mathcal{K}$ be a self-similar set for the i.f.s. $\{S_i\}_{i \in \mathcal{A}}$, and let s be the similarity dimension. Then $\mathcal{H}^s(\mathcal{K}) = 0$ if and only if for any $\varepsilon > 0$*

$$\exists u, v \in \mathcal{A}^*,\ u \neq v, \quad \|S_u^{-1} S_v - \mathrm{Id}\,\| < \varepsilon.$$

In the setting of Theorem 2.1 we have $S_i^\lambda(x) = r_i(\lambda)x + a_i(\lambda)$, so

$$\Big((S_u^\lambda)^{-1} S_v^\lambda - \mathrm{Id}\Big)(x) = \Big(\frac{r_v(\lambda)}{r_u(\lambda)} - 1\Big)(x - x_0(\lambda)) + \frac{S_v^\lambda(x_0(\lambda)) - S_u^\lambda(x_0(\lambda))}{r_u(\lambda)}.$$

Here $x_0(\lambda)$ is arbitrary; it is convenient to take $x_0(\lambda) = \Pi(\lambda, \bar{1})$ where $\bar{1} = 111\ldots \in \Omega$. Then $S_v^\lambda(x_0(\lambda)) = \Pi(\lambda, v\bar{1})$.

Definition: Let $\mathcal{V}_\varepsilon$ be the set of $\lambda \in J$ such that there exist distinct $u, v \in \mathcal{A}^*$ such that

$$\frac{r_v(\lambda)}{r_u(\lambda)} \in (e^{-\varepsilon}, e^{\varepsilon}) \tag{3.1}$$

and

$$|f_{v\bar{1},u\bar{1}}(\lambda)| = |\Pi(\lambda, v\bar{1}) - \Pi(\lambda, u\bar{1})| < \varepsilon r_u(\lambda). \tag{3.2}$$

Proof of Theorem 2.1(i): By Theorem 3.1, $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0$ if and only if $\lambda \in \mathcal{V}_\varepsilon$ for all $\varepsilon > 0$. Thus it is enough to prove that $\mathcal{IP} \setminus \mathcal{V}_\varepsilon$ has zero Lebesgue measure for any $\varepsilon > 0$. To this end, we will show that $\mathcal{IP} \setminus \mathcal{V}_\varepsilon$ has no Lebesgue density points. Fix $\varepsilon > 0$ and $\lambda_0 \in \mathcal{IP}$. Since $\lambda_0 \in \mathcal{IP}$, there exist $\omega, \tau \in \Omega$ such that $f_{\omega,\tau} \not\equiv 0$ but $f_{\omega,\tau}(\lambda_0) = 0$. Then $\mathcal{K}_{\omega|n}^{\lambda_0} \cap \mathcal{K}_{\tau|p}^{\lambda_0} \neq \emptyset$ for any $n, p \in \mathbb{N}$.

Now we outline the proof in a special case, to indicate the idea (which is inspired by [17]), and then provide all the details in the general case. Assume, for simplicity, that $r_i(\lambda) = r$ for all $i \leq m$ and all $\lambda \in J$. Let $u = \omega|n$ and $v = \tau|n$. The intersecting cylinders $\mathcal{K}_u^{\lambda_0}$ and $\mathcal{K}_v^{\lambda_0}$ have the same diameter $c_1 r^n$.

Transversality implies that (for n sufficiently large) these cylinders move relative to each other, as λ varies close to λ_0, with a speed that is uniformly bounded from both zero and infinity. Thus, within a distance $c_2 r^n$ from λ_0 there is an interval I of size $c_3 \varepsilon r^n$ such that $|\Pi(\lambda, v\bar{1}) - \Pi(\lambda, u\bar{1})| \le \varepsilon c_1 r^n$ for all $\lambda \in I$. This means that $I \subset \mathcal{V}_\varepsilon$, and since $|I|/c_2 r^n = c_3\varepsilon/c_2$ does not depend on n, we see that λ_0 is not a density point for $\mathcal{IP} \setminus \mathcal{V}_\varepsilon$. This concludes the outline of the proof in the special case.

To deal with the general case we use the following two lemmas. The first one is a standard result from renewal theory: for the proof see [7, Vol. II, Lemma 5.4.2].

LEMMA 3.2: *For any $\eta > 0$, $\beta \in (0,1)$, and $\lambda_0 \in J$ there exists $N \in \mathbb{N}$ with the following property: For any $s, t \in \mathcal{A}^*$, with $r_s(\lambda_0)/r_t(\lambda_0) \in [\beta, \beta^{-1}]$, there exist $u, v \in \mathcal{A}^*$ such that s, t are their respective prefixes, $|u| - |s| \le N$, $|v| - |t| \le N$, and*

$$\frac{r_u(\lambda_0)}{r_v(\lambda_0)} \in (e^{-\eta}, e^{\eta}).$$

LEMMA 3.3: *For any $u \in \mathcal{A}^*$ and $\lambda_1, \lambda_2 \in J$,*

$$\frac{r_u(\lambda_2)}{r_u(\lambda_1)} \le e^{\frac{L}{\beta}|u|\cdot|\lambda_2 - \lambda_1|} \tag{3.3}$$

where β is from (2.1) and $L := \max_i \|r_i'\|_{C(J)}$.

Proof: We have

$$\log \frac{r_i(\lambda_2)}{r_i(\lambda_1)} \le \frac{|r_i(\lambda_2) - r_i(\lambda_1)|}{r_i(\lambda_1)} \le \frac{L}{\beta}|\lambda_2 - \lambda_1|$$

for any $i \le m$ by (2.1), and (3.3) follows. ∎

Next we make some preliminary observations needed in the proof. For ζ and ξ in Ω let $\zeta \wedge \xi$ denote their largest common prefix (an empty word if $\zeta_1 \ne \xi_1$). It follows from (2.1) and (2.2) that

$$|f_{\zeta,\xi}(\lambda)| \le 2C_1 r_{\zeta\wedge\xi}(\lambda) \quad \text{for all } \zeta, \xi \in \Omega \tag{3.4}$$

where

$$C_1 = \max_i \|a_i\|_{C(J)} \cdot (1-\rho)^{-1}.$$

Further, by (2.3),

$$\{f_{\zeta,\xi}(\lambda) \colon \zeta, \xi \in \Omega\} \quad \text{is compact in } C^1(J). \tag{3.5}$$

Proof of Theorem 2.1(i) continued: Recall that $\varepsilon > 0$ and $\lambda_0 \in \mathcal{IP}$ are fixed and we are trying to prove that λ_0 is not a density point for $\mathcal{IP} \setminus \mathcal{V}_\varepsilon$. Let $\omega, \tau \in \Omega$ be such that $f_{\omega,\tau}(\lambda_0) = 0$ but $f_{\omega,\tau} \not\equiv 0$. Fix a large $k \in \mathbb{N}$ and find minimal n, p such that $r_{\omega|n}(\lambda_0) \le \rho^k$ and $r_{\tau|p}(\lambda_0) \le \rho^k$. Then $n, p \le k$ and $r_{\omega|n}(\lambda_0) r_{\tau|p}(\lambda_0) \in [\beta, \beta^{-1}]$ where β is defined in (2.1). By Lemma 3.2, there exist $u, v \in \mathcal{A}^*$ with $|u| - n \le N$, $|v| - p \le N$, such that $u|n = \omega|n$, $v|p = \tau|p$, and

$$\frac{r_u(\lambda_0)}{r_v(\lambda_0)} \in (e^{-\varepsilon/3}, e^{\varepsilon/3}).$$

Lemma 3.3 now implies that for

$$|\lambda - \lambda_0| \le \frac{\varepsilon\beta}{3L(k+N)}$$

we have

$$\frac{r_u(\lambda)}{r_u(\lambda_0)} \in (e^{-\varepsilon/3}, e^{\varepsilon/3}) \quad \text{and} \quad \frac{r_u(\lambda)}{r_v(\lambda)} \in (e^{-\varepsilon}, e^{\varepsilon}). \tag{3.6}$$

Next we find an interval of λ for which (3.2) holds. Since $f_{\omega,\tau}(\lambda_0) = 0$, it follows by the choice of u, v and (3.4) that

$$|f_{v\bar{1},u\bar{1}}(\lambda_0)| \le |f_{v\bar{1},\tau}(\lambda_0)| + |f_{u\bar{1},\omega}(\lambda_0)| \le 4C_1\rho^k. \tag{3.7}$$

We have $|f'_{\omega,\tau}(\lambda_0)| > \delta$ for some $\delta > 0$ by the transversality condition (2.4). By (3.5), we can find $\eta > 0$ and $k_0 \in \mathbb{N}$ so that

$$|\lambda - \lambda_0| < \eta \quad \text{and} \quad k \ge k_0 \Rightarrow |f'_{v\bar{1},u\bar{1}}(\lambda)| > \delta. \tag{3.8}$$

Denote $F_k := [\lambda_0 - 4C_1\rho^k/\delta, \lambda_0 + 4C_1\rho^k/\delta]$. If $k \ge k_0$ and $4C_1\rho^k/\delta < \eta$, then, in view of (3.7) and (3.8),

$$\exists \lambda_1 \in F_k \text{ such that } f_{v\bar{1},u\bar{1}}(\lambda_1) = 0. \tag{3.9}$$

We will show that λ_0 is not a density point for $\mathcal{IP} \setminus \mathcal{V}_\varepsilon$. To this end, we estimate $\mathcal{L}\{F_k \cap \mathcal{V}_\varepsilon\}$ from below. We can assume that $e^{\varepsilon/3} \le 2$ and k is so large that

$$\frac{4C_1\rho^k}{\delta} < \min\left\{\frac{\varepsilon\beta}{2L(k+N)}, \eta\right\}.$$

Then (3.1) holds on F_k by (3.6). Also by (3.6) we have

$$\beta^{N+1}\rho^k \le r_u(\lambda_0) \le 2r_u(\lambda) \quad \text{for } \lambda \in F_k.$$

Thus, for (3.2) to hold, it suffices that

$$|f_{v\bar{1},u\bar{1}}(\lambda)| \le (\varepsilon/2)\beta^{N+1}\rho^k. \tag{3.10}$$

It follows from (3.5) that there exists $C_2 > 0$ such that $\|f'_{\zeta,\xi}\|_{C(J)} \le C_2$ for all $\zeta,\xi \in \Omega$. Thus, $|f_{v\bar{1},u\bar{1}}(\lambda)| \le C_2|\lambda-\lambda_1|$, and the inequality (3.10) holds whenever

$$|\lambda - \lambda_1| \le \frac{\beta^{N+1}\rho^k\varepsilon}{2C_2}. \tag{3.11}$$

Choose ϵ small enough so that

$$\frac{\beta^{N+1}\rho^k\varepsilon}{C_2} \le \frac{4C_1\rho^k}{\delta}.$$

It follows from (3.9) and (3.11) that

$$\mathcal{L}\{F_k \cap \mathcal{V}_\varepsilon\} \ge |F_k| \cdot \frac{\beta^{N+1}\rho^k\varepsilon/(2C_2)}{8C_1\rho^k/\delta} = |F| \cdot \frac{\delta\beta^{N+1}\varepsilon}{16C_1C_2}.$$

Since $|F_k| \to 0$ as $k \to \infty$, we conclude that λ_0 is not a density point for $\mathcal{IP} \setminus \mathcal{V}_\varepsilon$. The proof is complete. ∎

Proof of Theorem 2.1(ii): Here (2.7) was assumed, hence $\mathcal{IP}$ is compact by Lemma 2.3(ii). Thus, the Baire Category Theorem holds in $\mathcal{IP}$. For $\varepsilon > 0$ let $\mathcal{V}_\varepsilon$ be as in the proof of part (i). Then

$$\{\lambda \in \mathcal{IP}\colon \mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0\} = \bigcap_{n\in\mathbb{N}}(\mathcal{V}_{1/n} \cap \mathcal{IP}).$$

It suffices to show that $\mathcal{V}_\varepsilon \cap \mathcal{IP}$ is dense in $\mathcal{IP}$ for any $\varepsilon > 0$ (it is immediate from (3.1) and (3.2) that $\mathcal{V}_\varepsilon$ is open). But this was, in fact, verified, in the course of the proof of part (i), since the point λ_1 from (3.9) is in $\mathcal{V}_\varepsilon \cap \mathcal{IP}$. The argument in part (i) can also be adapted to show that $\mathcal{IP}$ is perfect: Given $\lambda_0 \in \mathcal{IP}$ and $\omega,\tau \in \Omega$ such that $f_{\omega,\tau}(\lambda_0) = 0$ we may find, using the assumption (2.6), sequences $\tilde{\omega}$ and $\tilde{\tau}$ in Ω that agree with ω and τ respectively in the first n digits, and satisfy $f_{\tilde{\omega},\tilde{\tau}}(\lambda_0) \ne 0$. Then, using transversality, we find $\lambda_1 \in \mathcal{IP}$ close to λ_0 such that $f_{\tilde{\omega},\tilde{\tau}}(\lambda_1) = 0$ and, in particular, $\lambda_1 \in \mathcal{IP}$. ∎

Proof of Proposition 2.4: Say that $\lambda_0 \in \mathcal{IP}_\delta$ if there exist $\omega,\tau \in \Omega$ and $\delta > 0$ such that $f_{\omega,\tau}(\lambda_0) = 0$ and $|f'_{\omega,\tau}(\lambda_0)| \ge \delta$. By the transversality condition (2.4), we have $\mathcal{IP} = \bigcup_{n\in\mathbb{N}} \mathcal{IP}_{1/n}$, so it suffices to show that $\dim_H \mathcal{IP}_\delta \le 2s_{\max}$ for any fixed $\delta > 0$. For $u \in \mathcal{A}^*$ denote $I_u^\lambda = \mathsf{Conv}(\mathcal{K}_u^\lambda)$. If $\lambda_0 \in \mathcal{IP}_\delta$ and $\omega,\tau \in \Omega$ are as above, then the intervals I_u^λ and I_v^λ move relative to each other with a speed bounded below by $\delta/2$ as λ varies close to λ_0, provided that u, v are sufficiently long prefixes of ω, τ. (This follows from (3.5), as in the proof of Theorem 2.1(i).) Thus, the set $\{\lambda \in J\colon I_u^\lambda \cap I_v^\lambda \neq \emptyset\}$ is a union of at most $\frac{C}{\delta}$ intervals, each of which has length at most $\frac{C}{\delta}\max\{|I_u^\lambda| + |I_v^\lambda|\colon \lambda \in J\}$, with a uniform constant C.

Let us write $u \in \mathcal{C}_\varepsilon(\lambda)$ if $|I_u^\lambda| = r_u(\lambda) \leq \varepsilon$ and no prefix of u has this property. Since $\sum_{u \in \mathcal{C}_\varepsilon(\lambda)} r_u^{s(\lambda)} = 1$, the cardinality of $\mathcal{C}_\varepsilon(\lambda)$ is at most $\beta^{-1}\varepsilon^{-s(\lambda)}$. Let $J = [\lambda_1, \lambda_2]$. Observe that

$$|I_w^\lambda| \leq |I_w^{\lambda_1}|^t, \quad \text{where } t := \min_{\substack{i \leq m \\ \lambda \in J}} \frac{\log r_i(\lambda)}{\log r_i(\lambda_1)} \tag{3.12}$$

for all $\lambda \in J$ and $w \in \mathcal{A}^*$. It follows from the discussion above that the set $\mathcal{IP}_\delta$ may be covered by $\frac{C}{\delta}\beta^{-2}\varepsilon^{-2s(\lambda_1)}$ intervals each of length not greater than $\frac{2C}{\delta}\varepsilon^t$. Thus, $\dim_H \mathcal{IP}_\delta \leq 2s_1/t \leq 2s_{\max}/t$. Subdividing J into small intervals we can assume that t in (3.12) is arbitrarily close to one, hence $\dim_H \mathcal{IP}_\delta \leq 2s_{\max}$. ∎

4. Positive packing measure

THEOREM 4.1: *Let (Ω, d) be a complete separable metric space and $A \subset \Omega$ a Souslin subset with $\mathcal{H}^\gamma(A) > 0$. Suppose that we are given a one-parameter family of maps $\Pi_\lambda\colon A \to \mathbb{R}$, with $\lambda \in J$, where $J \subset \mathbb{R}$ is a closed interval. Assume that for some positive function $\alpha(\lambda)$ there exist positive δ and M such that for all $\omega \neq \tau$ the functions*

$$\Psi(\lambda) = \Psi_{\omega,\tau}(\lambda) = \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\tau)}{d(\omega, \tau)^{\alpha(\lambda)}}$$

belong to $C^1(J)$ and satisfy

$$\|\Psi\|_{C^1(J)} \leq M \tag{4.1}$$

and the transversality condition:

$$|\Psi(\lambda)| + |\Psi'(\lambda)| > \delta \quad \text{for all } \lambda \in J. \tag{4.2}$$

Then

$$\dim_H\{\lambda \in J\colon \mathcal{P}^{s(\lambda)}(\Pi_\lambda A) = 0\} \leq s_{\max}$$

where $s_{\max} = \sup\{s(\lambda)\colon \lambda \in J\}$ and $s(\lambda) = \gamma/\alpha(\lambda)$.

Remarks: (a) Since the functions Ψ are bounded, the maps Π_λ are Hölder with exponent $\alpha(\lambda)$, so $\mathcal{P}^\gamma(A) < \infty$ implies that $\mathcal{P}^{s(\lambda)}(\Pi_\lambda(A)) < \infty$.

(b) The statement of the theorem has content only when $s_{\max} < 1$.

The proof of the theorem is based on the following result on "projections" of a measure. Recall that for a Borel probability measure ν on $\mathbb{R}^n$ the lower α-dimensional density is defined by

$$\underline{D}_\alpha(\nu, x) = \liminf_{r \downarrow 0} \frac{\nu(B(x,r))}{r^\alpha}$$

where $B(x,r)$ is the closed ball of radius r centered at x.

THEOREM 4.2: *Suppose that we are in the setting of Theorem 4.1 and μ is a Borel probability measure on Ω such that*

$$(4.3) \qquad (\mu \times \mu)\{(\omega, \tau) \in \Omega^2 \colon d(\omega,\tau) < r\} \le Cr^\gamma \quad \textit{for all } r > 0.$$

Then the measure $\nu_\lambda = \Pi_\lambda \mu$ satisfies

$$(4.4) \qquad \dim_H\left\{\lambda \in J \colon \int \underline{D}_{s(\lambda)}(\nu_\lambda, x)\, d\nu_\lambda(x) = \infty\right\} \le s_{\max}.$$

In order to deduce Theorem 4.1 we use the following result:

THEOREM 4.3 (Taylor and Tricot [27, Thm. 5.4]): *For any $\alpha > 0$ and $n \in \mathbb{N}$ there exists a constant $p(\alpha, n) > 0$ with the following property: For any Borel probability measure ν on $\mathbb{R}^n$, Borel set $A \subset \mathbb{R}^n$ and $C > 0$,*

$$\underline{D}_\alpha(\nu, x) \le C \quad \textit{for all } x \in A \Rightarrow \mathcal{P}^\alpha(A) \ge C^{-1} p(\alpha, n) \nu(A).$$

Proof of the implication 4.2 $\Rightarrow$ 4.1: Since $\mathcal{H}^\gamma(A) > 0$, by Frostman's lemma in metric spaces, due to Howroyd [8] (see also [16, p. 120]), we can find a probability measure μ supported on A such that $\mu(B(x,r)) \le Cr^\gamma$ for any ball $B(x,r)$. Then, clearly, (4.3) is satisfied. The Borel probability measure $\nu_\lambda = \Pi_\lambda \mu$ is supported on $\Pi_\lambda A$ for all $\lambda \in J$. If $\underline{D}_{s(\lambda)}(\nu_\lambda, x) < \infty$ for ν_λ-a.e. x, then $\mathcal{P}^{s(\lambda)}(\Pi_\lambda A) > 0$ by Theorem 4.3. By (4.4), this happens for all $\lambda \in J$ except on a set of dimension less than or equal to $s_{\max}$, as desired. ∎

The proof of Theorem 4.2 relies on the following proposition, which will also be useful in Section 5.

PROPOSITION 4.4: *Under the assumptions of Theorem 4.2, let $s_1 > s_{\max}$ and let η be a probability (Frostman) measure on J such that $\eta(B(x,\rho)) \le c\rho^{s_1}$ for any $\rho > 0$. Then*

$$(4.5) \qquad \mathcal{J} := \int_J \int \nu_\lambda\left(B(x, r^{\alpha(\lambda)})\right) d\nu_\lambda(x) d\eta(\lambda) \le Cr^\gamma \quad \textit{for all } r > 0.$$

Proof of the implication 4.4 ⇒ 4.2: Suppose that the Hausdorff dimension of the set in (4.4) is greater than $s_{\max}$. Then Frostman's lemma implies that there exists a measure η as in Proposition 4.4, supported on this set. Since $s(\lambda) = \gamma/\alpha(\lambda)$, Fatou's Lemma and (4.5) yield that

$$\int_J \int \underline{D}_{s(\lambda)}(\nu_\lambda, x) d\nu_\lambda(\lambda) d\eta(\lambda) = \int_J \int \liminf_{r\downarrow 0} \frac{\nu_\lambda\Big(B(x, r^{\alpha(\lambda)})\Big)}{r^\gamma} d\nu_\lambda(x) d\eta(\lambda) < \infty,$$

which is a contradiction. ∎

The proof of Proposition 4.4 uses the following simple lemma.

LEMMA 4.5: *Suppose that $\Psi \in C^1(J)$ satisfies (4.1) and (4.2). Then for $\rho \le \delta/2$ the set $\{\lambda \in J\colon |\Psi(\lambda)| \le \rho\}$ is a union of at most $1 + M|J|/\delta$ intervals of length at most $4\rho/\delta$.*

Proof of Lemma 4.5: By (4.2), if $|\Psi(\lambda)| \le \rho \le \delta/2$ then $|\Psi'(\lambda)| \ge \delta/2$, so the set $\{\lambda \in J\colon |\Psi(\lambda)| \le \rho\}$ is a union of intervals on each of which the function Ψ is monotone. The length of each of these intervals is at most $2\rho/\delta/2 = 4\rho/\delta$. Moreover, each of these intervals lies in an interval of the set

$$\{\lambda \in J : \; |\Psi(\lambda)| \le \delta/2\},$$

and the latter intervals have length at least δ/M by (4.1). Since they don't intersect, their number is at most $1 + M|J|/\delta$. ∎

Proof of Proposition 4.4: Making a change of variable and exchanging the order of integration yields

$$\mathcal{J} = \int_\Omega \int_\Omega \eta\{\lambda \in J\colon |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \le r^{\alpha(\lambda)}\} d\mu(\omega) d\mu(\tau).$$

Decompose this integral as follows:

$$\mathcal{J} = \int_\Omega \int_\Omega = \int \int_{d(\omega,\tau)<r} + \sum_{k=1}^{\infty} \int \int_{2^{k-1}r \le d(\omega,\tau) < 2^k r} = \mathcal{J}_0 + \sum_{k=1}^{\infty} \mathcal{J}_k.$$

To estimate $\mathcal{J}_k$ recall that $\Psi_{\omega,\tau}(\lambda) = [\Pi_\lambda(\omega) - \Pi_\lambda(\tau)] d(\omega,\tau)^{-\alpha(\lambda)}$ and observe that for $d(\omega,\tau) \ge 2^{k-1}r$ we have

$$\begin{aligned}\eta\{\lambda\colon |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \le r^{\alpha(\lambda)}\} &\le \eta\{\lambda\colon |\Psi_{\omega,\tau}(\lambda)| \le r^{\alpha(\lambda)}(2^{k-1}r)^{-\alpha(\lambda)}\} \\ &= \eta\{\lambda\colon |\Psi_{\omega,\tau}(\lambda)| \le 2^{-(k-1)\alpha(\lambda)}\} \\ &\le \eta\{\lambda\colon |\Psi_{\omega,\tau}(\lambda)| \le 2^{-(k-1)\alpha_{\min}}\}\end{aligned}$$

where $\alpha_{\min} = \inf_{\lambda \in J} \alpha(\lambda)$. Choose $k_0 \in \mathbb{N}$ such that $2^{-(k_0-1)\alpha_{\min}} \leq \delta/2$. Fix $k \geq k_0$ and $\rho = 2^{-(k-1)\alpha_{\min}}$. Let $\Psi = \Psi_{\omega,\tau}$. Since η is a Frostman measure, we obtain for $k \geq k_0$ by Lemma 4.5:

$$\eta\{\lambda\colon |\Psi_{\omega,\tau}(\lambda)| \leq 2^{-(k-1)\alpha_{\min}}\} \leq C2^{-k\alpha_{\min}s_1} \tag{4.6}$$

where the constant C does not depend on k, ω, or τ. Now we continue the estimate of $\mathcal{J}$ using (4.6) and (4.3):

$$\begin{aligned}\mathcal{J} \leq & \sum_{k=0}^{k_0-1} (\mu \times \mu)\{(\omega,\tau)\colon d(\omega,\tau) < 2^k r\} \\ & + \sum_{k=k_0}^{\infty} 2^{-k\alpha_{\min}s_1}(\mu \times \mu)\{(\omega,\tau)\colon d(\omega,\tau) < 2^k r\} \\ \leq & C' \sum_{k=0}^{k_0-1} (2^k r)^\gamma + C' \sum_{k=k_0}^{\infty} 2^{-k\alpha_{\min}s_1}(2^k r)^\gamma \\ \leq & C''\big(2^{k_0} + 2^{k_0(\gamma-\alpha_{\min}s_1)}\big) r^\gamma.\end{aligned}$$

In the last inequality we used the hypothesis that $s_1 > s_{\max} = \gamma/\alpha_{\min}$. The proof is complete. ∎

Proof of Theorem 2.1 (iii): We let $\Omega = \mathcal{A}^{\mathbb{N}}$, as in Section 3, and equip it with the metric $d(\omega,\tau) = r_{\omega\wedge\tau}(\lambda_1)$ where $J = [\lambda_1, \lambda_2]$. Further, let $A = \Omega$ and $\Pi_\lambda = \Pi(\lambda,\cdot)$ be the natural projection map (2.2). Since $r_i(\lambda) = r_i^{\varphi(\lambda)}$, (3.4) implies that Π_λ is Hölder with exponent $\alpha(\lambda) = \varphi(\lambda)/\varphi(\lambda_1)$. Then

$$\Psi_{\omega,\tau}(\lambda) = \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\tau)}{d(\omega,\tau)^{\alpha(\lambda)}} = \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\tau)}{r_{\omega\wedge\tau}(\lambda)} = \Pi_\lambda(\omega') - \Pi_\lambda(\tau')$$

where $\omega_1' \neq \tau_1'$. Since (2.7) is assumed, the transversality condition (4.2) holds by Lemma 2.3(iv), and (4.1) is obviously satisfied. We have $\dim_H A = \gamma = s(\lambda_1)$, with $\mathcal{H}^\gamma(A) > 0$. By the definition of the similarity dimension we have $s(\lambda_1)/\alpha(\lambda) = s(\lambda)$, and the claim follows by Theorem 4.1. ∎

Proof of Proposition 1.3: We let (Ω, d) be $\mathbb{R}^2$ with the Euclidean metric. Further, $A = \mathcal{K}$ and $\Pi_\theta = \mathrm{proj}_\theta$. Then $\alpha(\theta) = 1$ for all θ and conditions (4.1), (4.2) are obviously satisfied. Thus, the claim follows by Theorem 4.1. ∎

5. Positive capacity in general kernels

Let $\Phi \in C^1(0,\infty)$ be a positive decreasing function which vanishes for all x sufficiently large. Recall the following (see [2]):

Definition: Let ν be a Borel probability measure on $\mathbb{R}$. The **Φ-energy** of ν is defined by

$$\mathcal{E}_\Phi(\nu) = \int\int \Phi(|x-y|)\, d\nu(x) d\nu(y).$$

The **Φ-capacity** of a Souslin set $F \subset \mathbb{R}$ is defined by

$$\mathrm{Cap}_\Phi(F) = \sup\{\mathcal{E}_\Phi(\nu)^{-1} \colon \nu = \text{ Borel probability on } F\}.$$

THEOREM 5.1: *Suppose that we are in the setting of Theorem 4.1, Φ is a kernel as above, and let $\Phi_\lambda(r) = \Phi(r^{1/\alpha(\lambda)})$ for $\lambda \in J$.*

(i) *If $\int_0^\infty r^\gamma |\Phi'(r)| dr < \infty$ and $\mathcal{H}^\gamma(A) > 0$ then*

$$\dim_H\Big\{\lambda \in J \colon \mathrm{Cap}_{\Phi_\lambda}(\Pi_\lambda A) = 0\Big\} \leq s_{\max}. \tag{5.1}$$

(ii) *If $\int_0^\infty r^\gamma |\Phi'(r)| dr = \infty$ and $\mathcal{P}^\gamma(A) < \infty$ then $\mathrm{Cap}_{\Phi_\lambda}(\Pi_\lambda A) = 0$ for all $\lambda \in J$.*

Proof of Theorem 5.1: (i) As in the proof of Theorem 4.1, let μ be the Frostman measure supported on A with exponent γ and let $\nu_\lambda = \Pi_\lambda \mu$. Then

$$\begin{aligned} \mathcal{E}_{\Phi_\lambda}(\nu_\lambda) &= \int\int \Phi(|x-y|^{1/\alpha(\lambda)}) d\nu_\lambda(x) d\nu_\lambda(y) \\ &= \int_0^1 (\nu_\lambda \times \nu_\lambda)\{(x,y)\colon |x-y| \leq r^{\alpha(\lambda)}\} |\Phi'(r)|\, dr. \end{aligned} \tag{5.2}$$

Suppose that the set in (5.1) has dimension $s_1 > s_{\max}$ and let η be the Frostman measure on this set satisfying $\eta(I) \leq c|I|^{s_1}$ for any interval $I \subset J$. We have

$$\begin{aligned} \int_J \mathcal{E}_{\Phi_\lambda}(\nu_\lambda)\, d\eta(\lambda) = & \int_0^\infty \int_J \int \nu_\lambda(B(x, r^{\alpha(\lambda)})) |\Phi'(r)| d\nu_\lambda(x) d\eta(\lambda) dr \\ \leq & C \int_0^\infty r^\gamma |\Phi'(r)| dr < \infty \end{aligned}$$

by the Fubini Theorem, (5.2), and Proposition 4.4. Thus, $\mathrm{Cap}_{\Phi_\lambda}(\Pi_\lambda A) > 0$ for η-a.e. λ, a contradiction.

(ii) Note that Π_λ is $\alpha(\lambda)$-Hölder for all $\lambda \in J$. Fix any $\lambda \in J$. By the definition of packing measure, see [16], there is a countable covering of A by sets of finite prepacking measure. Since capacity is countably subadditive, it is enough to prove the statement for each of these sets. Thus, without loss of generality, A can be covered by $Cr^{-\gamma}$ balls of radius r for all $r > 0$, hence $\Pi_\lambda A$ can be covered by $C\rho^{-s(\lambda)}$ balls of radius ρ for all $\rho > 0$. Changing the variable $r = \rho^{1/\alpha(\lambda)}$ we have

$$\int_0^\infty \rho^{s(\lambda)}|\Phi'_\lambda(\rho)|d\rho = \int_0^\infty r^\gamma|\Phi'(r)|dr = \infty.$$

By [2, Thm. IV.2], this yields $\mathrm{Cap}_{\Phi_\lambda}(\Pi_\lambda A) = 0$. ∎

6. Further examples

Example 6.1: Let $\mathcal{C}_r^2 = \mathcal{C}_r \times \mathcal{C}_r$ where $\mathcal{C}_r$ is the middle-α Cantor set for $\alpha = 1 - 2r$. The family of projections $\{\mathrm{proj}_\theta \mathcal{C}_r^2\}_{0\le\theta<\pi}$ is affine-equivalent to the family of self-similar sets for the i.f.s. $\{rx, rx+1, rx+u, rx+1+u\}_{u>0}$. We assume that $r < \frac{1}{4}$ so $\dim_H \mathcal{C}_r^2 < 1$. The analysis of this family is similar to the one done in Example 2.8. The transversality condition (2.4) follows from Lemma 2.6. To apply Lemma 2.5 we check (2.11) and (2.12). One can check that $\mathcal{IP} \supset F := [\arctan\frac{1-2r}{r}, \arctan\frac{2}{1-3r}]$, which is a non-empty interval for $r \in (\frac{1}{6}, \frac{1}{4})$. The conclusion is that for all $r \in (\frac{1}{6}, \frac{1}{4})$ and a.e. $\theta \in F$,

$$\mathcal{H}^s(\mathrm{proj}_\theta \mathcal{C}_r^2) = 0 \quad \text{and} \quad 0 < \mathcal{P}^s(\mathrm{proj}_\theta \mathcal{C}_r^2) < \infty, \quad \text{where } s = \log 4/|\log r|.$$

Since $\mathcal{C}_r^2 - \mathcal{C}_r^2$ is a planar self-similar set having similarity dimension $\log 9/|\log r|$ we have $\mathcal{L}(\mathcal{IP}) = 0$ for all $r < \frac{1}{9}$ by Lemma 2.7. The case $r \in (\frac{1}{9}, \frac{1}{6})$ remains open although we suspect that $\mathcal{L}(\mathcal{IP}) > 0$ for a.e. such r.

Remark: For $s = 1$ the statement of Theorem 1.2 still holds, and is well-known. In Examples 2.8 and 6.1 this corresponds to $r = \frac{1}{3}$ and $r = \frac{1}{4}$ respectively. In this case more precise information is available: the one-dimensional Hausdorff measure is zero for all projections in irrational directions. This was proved by Kenyon [13] and Lagarias and Wang [14].

Non-homogeneous families (see the definition in Section 2). A variant of Lemma 2.5 holds for such families, provided (2.9), (2.10) are replaced by

(a) $\mathsf{Conv}(K_i^\lambda)$ intersects K_j^λ for all $i \neq j$ and $\lambda \in \widetilde{J}$;

(b) the *Newhouse thickness* of $\mathcal{K}^\lambda$ is greater than one for $\lambda \in \widetilde{J}$.

(See [19] for the definition of Newhouse thickness.) However, condition (b) is not easy to check (note also that in the homogeneous case (b) is more restrictive than (2.10)). Still, the notion of thickness is used to verify that $\mathcal{L}(\mathcal{IP}) > 0$ in the following example; this example is inspired by Moreira [18].

Example 6.2: Fix $\alpha, \beta, \gamma > 0$ so that $\min\{\alpha, \beta\} > \frac{1}{3}$ and $\alpha + \beta + \gamma < 1$. Consider the self-similar set $\mathcal{K} \subset \mathbb{R}^2$ defined by the i.f.s. $\{S_i\}_{i \leq 3}$, where $S_1(\mathbf{x}) = \alpha\mathbf{x}$, $S_2(\mathbf{x}) = \beta\mathbf{x} + (1,0)$, and $S_3(\mathbf{x}) = \gamma\mathbf{x} + (0,1)$. Clearly, the Strong Separation Condition is satisfied and $s = \dim_H \mathcal{K} < 1$. We are interested in the family of projections of $\mathcal{K}$ which is affine-equivalent to the family $\mathcal{K}^u$ of self-similar sets for the i.f.s. $\{\alpha x, \beta x + 1, \gamma x + u\}_{u \in \mathbb{R}}$ on the real line. Transversality (2.4) for the family of projections holds by Lemma 2.6. To check when $\mathcal{L}(\mathcal{IP}) > 0$ observe that $\mathcal{K}^u \supset \mathcal{K}_{\alpha\beta}$ where $\mathcal{K}_{\alpha\beta}$ is the self-similar set for the i.f.s. $\{\alpha x, \beta x + 1\}$. Since $\min\{\alpha, \beta\} > \frac{1}{3}$ we have that $\mathcal{K}_{\alpha\beta}$ is a Cantor set of thickness greater than one. Let $\mathcal{K}_1^u = \alpha\mathcal{K}^u$ and $\mathcal{K}_2^u = \gamma\mathcal{K}^u + u$. By the Gap Lemma of Newhouse (see [19]),

$$\mathcal{K}_1^u \cap \mathcal{K}_2^u \supset \alpha\mathcal{K}_{\alpha\beta} \cap (\gamma\mathcal{K}_{\alpha\beta} + u) \neq \emptyset$$

whenever $(\alpha\mathcal{K}_{\alpha\beta}) \cap \mathsf{Conv}(\gamma\mathcal{K}_{\alpha\beta} + u) \neq \emptyset$. Since $\mathsf{Conv}(\mathcal{K}_{\alpha\beta}) = [0, \frac{1}{1-\beta}]$, the last condition certainly holds for $u \in F := [-\frac{\gamma}{1-\beta}, 0]$. Theorem 2.1 implies that $\mathcal{H}^s(\mathrm{proj}_\theta \mathcal{K}) = 0$ and $0 < \mathcal{P}^s(\mathrm{proj}_\theta \mathcal{K}) < \infty$ for a.e. θ with $\arctan\theta \in F$.

7. Generalizations and concluding remarks

Multidimensional generalizations. All the statements of the paper extend to higher dimensions but we restrict attention to the main results, emphasizing the case of projections. Of course, we need transversality conditions. For simplicity, we state them in the form most convenient for the proof.

We consider similitudes in $\mathbb{R}^\ell$ which do not involve any rotations, that is, $S_i^\lambda(\mathbf{x}) = r_i(\lambda)\mathbf{x} + a_i(\lambda)$ for $i \leq m$ and $\lambda \in J$. Assume that $J \subset \mathbb{R}^p$ is open, with $p \geq \ell$, the condition (2.1) holds, $r_i \in C^1(\mathrm{clos}\, J)$, and $a_i \in C^1(\mathrm{clos}\ J \to \mathbb{R}^\ell)$. The set-up at the beginning of Section 2 readily extends to this situation (including the definitions of $\Pi(\lambda, \omega)$, $f_{\omega,\tau}$, and $\mathcal{IP}$). The only modification needed is in the definition of transversality. We write $\mathcal{L}_p$ for the Lebesgue measure in $\mathbb{R}^p$.

Say that the p-parameter family of i.f.s. $\{S_1^\lambda, \ldots, S_m^\lambda\}_{\lambda \in J}$ satisfies the transversality condition if for any $\omega \neq \tau$ in Ω such that $f_{\omega,\tau} \not\equiv \mathbf{0}$, there exist C_1, C_2 such that for all $r, \varepsilon > 0$ and $\lambda_0 \in J$,

$$\text{if } |f_{\omega,\tau}(\lambda_0)| < r \text{ then}$$

$$\mathcal{L}_p\Big\{\lambda \in B(\lambda_0, C_1 r) \colon |f_{\omega,\tau}(\lambda_0)| < \varepsilon r\Big\} \geq C_2 \varepsilon^\ell \mathcal{L}_p(B(\lambda_0, C_1 r)). \tag{7.1}$$

THEOREM 7.1: *Suppose that the family of i.f.s. satisfies (2.1) and (7.1). If $\mathcal{IP} \neq \emptyset$, then*

(i) $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0$ *for a.e.* $\lambda \in \mathcal{IP}$.

(ii) *If, in addition, (2.7) holds, then* $\{\lambda \in \mathcal{IP}\colon \mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0\}$ *is a dense* G_δ *set in* $\mathcal{IP}$.

Recall that $G(n,\ell)$ is the Grassmann manifold of all ℓ-planes in $\mathbb{R}^n$ passing through the origin and $\dim G(n,\ell) = \ell(n-\ell)$, see [16]. Denote by proj_Θ the orthogonal projection of $\mathbb{R}^n$ on $\Theta \in G(n,\ell)$.

COROLLARY 7.2: *Let* $\mathcal{K} = \bigcup_{i\le m}(r_i\mathcal{K} + a_i)$, *with* $r_i \in (0,1)$ *and* $a_i \in \mathbb{R}^n$, *be a self-similar set in* $\mathbb{R}^n$, *and let* s *be the similarity dimension.*

(i) *If the OSC holds, then* $\mathcal{H}^s(\mathrm{proj}_\Theta \mathcal{K}) = 0$ *for a.e.* $\Theta \in \mathcal{IP}$.

(ii) *If the Strong Separation Condition holds, then* $\{\Theta \in \mathcal{IP}\colon \mathcal{H}^s(\mathrm{proj}_\Theta \mathcal{K}) = 0\}$ *is a dense* G_δ *set in* $\mathcal{IP}$.

One can show that in the setting of Corollary 7.2,

$$\mathcal{IP} = \{\Theta \in G(n,\ell)\colon \mathrm{proj}_\Theta|_{\mathcal{K}} \text{ is not one-to-one}\}.$$

Next we state the analog of Theorem 4.1.

THEOREM 7.3: *Let* (Ω, d) *be a complete separable metric space and* $A \subset \Omega$ *a Souslin subset with* $\mathcal{H}^\gamma(A) > 0$. *Suppose that we are given a one-parameter family of maps* $\Pi_\lambda\colon A \to \mathbb{R}^\ell$, *with* $\lambda \in J$, *where* $J \subset \mathbb{R}^p$ *is open and* $p \ge \ell$. *Assume that for some function* $\alpha(\lambda)$ *there exist* $M, C > 0$ *such that for all* $\omega \neq \tau$ *the functions*

$$\Psi(\lambda) = \Psi_{\omega,\tau}(\lambda) = \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\tau)}{d(\omega,\tau)^{\alpha(\lambda)}}$$

belong to $C^1(\mathrm{clos}\, J)$, *with* $\|\Psi\|_{C^1} \le M$, *and satisfy the transversality condition: for all* $r > 0$,

$$\{\lambda \in J\colon \|\Psi(\lambda)\| < r\} \text{ can be covered by } Cr^{\ell-p} \text{ balls of radius } r. \tag{7.2}$$

Then

$$\dim_H\{\lambda \in J\colon \mathcal{P}^{s(\lambda)}(\Pi_\lambda A) = 0\} \le s_{\max} + (p-\ell)$$

where $s(\lambda) = \gamma/\alpha(\lambda)$.

COROLLARY 7.4: *Let* $\mathcal{K} \subset \mathbb{R}^n$ *be any Souslin set such that* $\mathcal{H}^s(\mathcal{K}) > 0$ *for some* $s \in (0,\ell)$. *Then* $\mathcal{P}^s(\mathrm{proj}_\Theta \mathcal{K}) > 0$ *for a.e.* $\Theta \in G(n,\ell)$. *Moreover,* $\dim_H\{\Theta \in G(n,\ell)\colon \mathcal{P}^s(\mathrm{proj}_\Theta \mathcal{K}) = 0\} \le \ell(n-\ell) + (s-\ell)$.

Outline of the proof of Theorem 7.1: Let us restrict attention to the measure-theoretic statement in Theorem 7.1(i); the topological statement follows the same general scheme, but is easier. The argument follows the proof of Theorem 2.1(i), so we refer to it without repeating all the notation. Since contraction rates $r_i(\lambda)$ are scalar, the beginning of the proof requires no change. The point where distinctions arise is after (3.7) where the transversality condition was applied. Now we use transversality (7.1) with $r = C\rho^k$ and see that λ_0 is not a Lebesgue density point. ∎

Proof of Corollary 7.2: Here the parameter set J is a local coordinate chart for the Grassmann manifold $G(n,\ell)$, with $p = \dim G(n,\ell) = \ell(n-\ell)$. As a metric d on $G(n,\ell)$ we take the Hausdorff metric for the intersections of ℓ-planes with the unit sphere in $\mathbb{R}^n$. Let $\Pi_\Theta(\omega)$ be the orthogonal projection of $\widetilde{\Pi}(\omega) \in \mathcal{K} \subset \mathbb{R}^n$ on the plane $\Theta \in J$ where $\widetilde{\Pi}$ is the natural projection map from the sequence space Ω to $\mathcal{K}$. We need to check the transversality condition (7.1). Observe that $f_{\omega,\tau}(\Theta)$ is the orthogonal projection of the vector $\mathbf{x} := \widetilde{\Pi}(\omega) - \widetilde{\Pi}(\tau) \neq \mathbf{0}$ on $\Theta \in J$. If $|f_{\omega,\tau}(\Theta_0)| < r$, then there exists an ℓ-plane Θ_1 orthogonal to $\mathbf{x}$, such that $d(\Theta_0,\Theta_1) \le C_1 r$. Now it is easy to see that the set

$$\{\Theta \in B(\Theta_0, C_1 r)\colon |f_{\omega,\tau}(\Theta)| < \varepsilon r\}$$

contains an (εr)-neighborhood of a manifold of dimension $\dim G(n-\ell,\ell) = \ell(n-\ell-1)$ intersected with $B(\Theta_0, C_1 r)$. This implies

$$\begin{aligned}\mathcal{L}_p\{\Theta \in B(\Theta_0, C_1 r)\colon |f_{\omega,\tau}(\Theta)| < \varepsilon r\} &\ge C(\varepsilon r)^\ell r^{\ell(n-\ell-1)} \\ &\ge C'\varepsilon^\ell \mathcal{L}_p(B(\Theta_0, C_1 r)),\end{aligned}$$

and (7.1) is verified. Thus, Theorem 7.1 can be applied and the claim follows. ∎

Proof of Theorem 7.3: The proof of Theorem 4.1 transfers, except that instead of the application of Lemma 4.5 in (4.6), we refer to the transversality condition (7.2). ∎

Proof of Corollary 7.4: As in the proof of Corollary 7.2, the parameter set J is a local coordinate chart for the Grassmann manifold $G(n,\ell)$, with $p = \ell(n-\ell)$. Further, $\Pi_\Theta(\omega)$ is the orthogonal projection of $\omega \in \mathcal{K} = A \subset \mathbb{R}^n$ on the plane $\Theta \in J$ and $\alpha(\Theta) = 1$ for all $\Theta \in J$. The only issue which requires discussion is the transversality condition (7.2). By definition, $\Psi(\Theta)$ is the orthogonal projection of a unit vector in $\mathbb{R}^n$ to Θ. It is easy to see that $\{\Theta \in G(n,\ell)\colon |\Psi(\Theta)| < r\}$ lies in a cr-neighborhood of a smooth manifold of dimension $\dim G(n-1,\ell) = \ell(n-\ell-1)$

in $G(n,\ell)$ and is therefore contained in the union of no more that $C(r^{-1})^{\ell(n-\ell-1)}$ balls of radius r. This proves (4.2) and Theorem 7.3 may be applied. ∎

Open Questions. In the first three questions, we restrict attention to orthogonal projections of a self-similar set $\mathcal{K}$ in the plane that satisfies the OSC and has dimension $s \in (0,1)$.

(i) Is it true that $\mathcal{P}^s(\mathrm{proj}_\theta \mathcal{K}) > 0$ for a residual set of θ?

(ii) We have shown that in many cases $\mathcal{H}^s(\mathrm{proj}_\theta \mathcal{K}) = 0$ for a typical θ in some interval. Is there a gauge function φ such that $\mathcal{H}^\varphi$ is the natural measure (up to scaling) for such projections?

(iii) Find a bound on the dimension (Hausdorff or packing) of the exceptional set in Theorem 1.2(i) on Hausdorff measure of projections.

(iv) Find a specific self-similar set of zero Hausdorff measure and positive packing measure.

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